Dispersion and birefringence in a synchrotron-emitting gas

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The response tensor for a highly relativistic, magnetized electron gas is evaluated by performing the integrals over gyrophase and pitch angle using the method of stationary phase. Only the transverse components are considered explicitly. The anti-Hermitian part is expressed in terms of the Airy function Ai(z), and its derivative and integral, and is shown to reproduce the known formulas for synchrotron absorption. It is shown that the full tensor is obtained by the replacement Ai$(z)\rightarrow$Ai$(z)+i$Gi(z), where Gi(z) is a generalized Airy function. A simple form for the high-frequency response is derived for a power-law distribution of particles. [S1063-651X(97)03309-6]

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I. INTRODUCTION

A distribution of highly relativistic electrons (Lorentz factor $\gamma\gg 1$) in a magnetic field emits and absorbs through the synchrotron process. Such a synchrotron-emitting gas may be regarded as a highly relativistic plasma, with its own characteristic dispersion and birefringence. The properties of the natural modes of such a medium influence the polarization of radiation passing through it; if the natural modes are linearly polarized they can cause a partial conversion of linear into circular polarization, as in a quarter-wave plate. The circularly polarized component of the radiation from some synchrotron sources is small but observable. There is a small intrinsic component of circular polarization but its frequency dependence is not consistent with the simplest interpretation of the data [1]. A possible alternative explanation, which is the motivation for the present investigation, is partial conversion of linear into circular polarization as a propagation effect. Although the response tensor for a synchrotron-emitting gas was investigated in this connection, the existing treatments are unsatisfactory: Ref. [2] used a method that failed to preserve the analytic relation between the Hermitian and the anti-Hermitian parts (implied by the causal condition), and this invalidates the important check that the anti-Hermitian part reproduces the known formulas for synchrotron absorption; Ref. [3] argued incorrectly that the natural modes are circularly polarized, and hence failed to treat the terms that imply the dominant linear polarization correctly.

In this paper an alternative method is used to derive the response tensor for a synchrotron-emitting gas. The method is based on the properties of synchrotron emission. Relativistic beaming implies that the radiation received by a distant observer is only from particles with pitch angle α within $O(\gamma^{-1})$ of the angle θ of the line of sight, and within a gyrophase angle $O(\gamma^{-1})$ of the phase at which the particle is moving directly toward the observer. As a consequence, integrals over gyrophase and pitch angle, in the general expression for response tensor, are dominated by contributions from ranges of these variables of $O(\gamma^{-1})$. Such integrals are well suited to evaluation by the method of the stationary phase. It is found here that this method preserves the analytic properties of the response tensor, and and it is shown that the anti-Hermitian part reproduces the known formulas for synchrotron absorption exactly. The resulting expression for the response tensor is applied to a power-law distribution of relativistic particles.

II. THE RESPONSE TENSOR

In a standard 4-tensor notation [4], the linear response is described by the relation [5] $J^{\mu}(k)=\alpha^{\mu}{}_{\nu}(k)A^{\nu}(k)$ between the induced 4-current, $J(k)$ and the 4-potential, $A(k)$, as functions of $k=(\omega,\mathbf{k})$. The particles are described by their distribution $F(p)$ in eight-dimensional phase space [6]. Two alternative expressions for the response tensor in this notation are derived using a forward scattering method and the Vlasov method, respectively. Both may be written in terms of integrals along the orbits of the particles.

The orbit of a particle is described by $x=X(\tau)$ with $x^{\mu}=(t,\mathbf{x})$ (units with $c=1$) so that the 4-velocity is $u(\tau)=\dot{X}(\tau)$, where the dot denotes differentiation with respect to the proper time τ. In a magnetostatic field $F_0^{\mu\nu}=Bf^{\mu\nu}$, $B=1/2F_0^{\mu\nu}F_{0\mu\nu}$, the orbit may be written as

$$X^{\mu}(\tau)=x_0^{\mu}+t^{\mu\nu}(\tau)u_{0\nu},\quad u^{\mu}(\tau)=\dot{t}^{\mu\nu}(\tau)u_{0\nu},\tag{1}$$

where x_0 and u_0 describe the initial conditions. Solving the equation of motion $m\dot{u}^{\mu}(\tau)=qF_0^{\mu\nu}u_{\nu}(\tau)$ for a particle with charge q and rest mass m gives

$$t^{\mu\nu}(\tau)=g_{\parallel}^{\mu\nu}\tau+g_{\perp}^{\mu\nu}\frac{\sin\Omega_0\tau}{\Omega_0}-\eta f^{\mu\nu}\frac{\cos\Omega_0\tau}{\Omega_0},\tag{2}$$

with $\Omega_0=|q|B/m$, $\eta=q/|q|$, $g_{\perp}^{\mu\nu}=-f^{\mu\alpha}f_{\alpha}{}^{\nu}$, and $g_{\parallel}^{\mu\nu}=g^{\mu\nu}-g_{\perp}^{\mu\nu}$.

In this notation, the forward scattering method gives

$$\alpha^{\mu\nu}(k)=\frac{q^2}{m}\int d^4p(\tau)F(p)\int_0^{\infty}d\xi\ \exp\{ik[X(\tau)-X(\tau-\xi)]\}$$
$$\times T_{\alpha\beta}(\xi)ku(\tau)\ G^{\alpha\mu}(k,u(\tau))ku(\tau-\xi)$$
$$\times G^{\beta\nu}(k,u(\tau-\xi)),\tag{3}$$

$$T^{\mu\nu}(\xi)=t^{\mu\nu}(\xi)-t^{\mu\nu}(0),\tag{4}$$

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$$G^{\mu\nu}(k,u)=g^{\mu\nu}-\frac{k^\mu u^\nu+k^\nu u^\mu}{ku}+\frac{k^2\ u^\mu u^\nu}{(ku)^2}, \tag{5}$$

and the Vlasov method gives

$$\alpha^{\mu\nu}(k)=-iq^2\int d^4p(\tau)\int_0^\infty d\xi\ u^\mu(\tau)$$
$$\times e^{ik[X(\tau)-X(\tau-\xi)]}[ku(\tau-\xi)g^{\alpha\nu}-k^\alpha u^\beta(\tau-\xi)]$$
$$\times \dot{t}_\alpha{}^\beta(\tau-\xi)\ \frac{\partial F(p)}{\partial p^\beta}, \tag{6}$$

where the integrals over $p(\tau)=mu(\tau)$ depend on τ through the gyrophase $\phi=\phi_0+\Omega_0\tau$. The two forms (3) and (6) are related by a partial integration. The final expressions derived below from them are similarly connected by a partial integration, which provides a useful check on the results.

Choosing a specific frame, referred to as the plasma frame, the integral in Eq. (6) may be written as over $dp^0 d\phi\ d\cos\alpha\ d|\mathbf{p}||\mathbf{p}|^2$, where α is the pitch angle. The integral over either p^0 or $|\mathbf{p}|$ may be performed over the δ function in the relation, $F(p)=2m\delta(p^2-m^2)f(\mathbf{p})$, between $F(p)$ and the distribution function $f(\mathbf{p})$ in six-dimensional phase space. The energy spectrum $N(\gamma)$ is also used below, and the relations between the various distributions are

$$F(p)=2mn\delta(p^2-m^2)f(|\mathbf{p}|)\phi(\alpha),$$
$$N(\gamma)=4\pi m^3\gamma^2 v\ f(\gamma m v), \tag{7}$$

$$4\pi\int_0^\infty d|\mathbf{p}|\ |\mathbf{p}|^2 f(|\mathbf{p}|)=\int_1^\infty d\gamma\ N(\gamma)=n,$$

$$\frac{1}{2}\int_{-1}^{1} d\cos\ \alpha\ \phi(\alpha)=1, \tag{8}$$

where n is the number density in the plasma frame, and $\phi(\alpha)$ the pitch angle distribution. (The assumption that the distribution is separable may be relaxed simply at the expense of complicating the notation.) Choosing to integrate over $|\mathbf{p}|=\gamma m v$, and writing $p^0=m\gamma$, the derivative in Eq. (6) becomes

$$ku(\tau)G^{\alpha\nu}(k,u(\tau))\dot{t}_\alpha{}^\beta(\tau)\ \frac{\partial}{\partial p^\beta}$$
$$=[ku(\tau)\ \tilde{u}^\nu-k\tilde{u}\ u^\nu(\tau)]\frac{1}{m\gamma}\frac{\partial}{\partial\gamma}$$
$$-[ku(\tau)a^\nu(\tau)-ka(\tau)u^\nu(\tau)]\frac{1}{m\gamma^2 v^2}\frac{\partial}{\partial\alpha}, \tag{9}$$

with $a^\mu(\tau)=d[u^\mu(\tau)]/d\alpha$, and where $\tilde{u}$ is the 4-velocity of the plasma frame.

In the following only the transverse component of the response tensor (3) or (6) is considered. The transverse plane is chosen to be the 12-plane, with the 1-axis along the component of the magnetic field, **B**, orthogonal to **k**, and the 2-axis orthogonal to both **B** and **k**. The integrands in both (3) and (6) depend on the proper times τ and ξ through the phase factor, and through $u^\mu(\tau)$, $u^\nu(\tau-\xi)$, $ku(\tau)$, and $ku(\tau-\xi)$. These various terms are expanded in powers of $1/\gamma$, and only the lowest order terms are retained. The specific terms that appear in the integrands in (3) and (6) are summarized in Appendix A.

III. EVALUATION BY STATIONARY PHASE

The resulting integrals over gyrophase and pitch angle in Eqs. (3) and (6) may be written in the generic form

$$\langle K\rangle(\xi)=\frac{1}{4\pi\phi(\theta)}\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\alpha\,\phi(\alpha)$$
$$\times K(\phi,\alpha,\xi)e^{ik[X(\tau)-X(\tau-\xi)]}, \tag{10}$$

with $\phi=\phi_0+\Omega_0\tau$. The phase factor in Eqs. (3) and (6) is

$$k[X(\tau)-X(\tau-\xi)]$$
$$=\frac{1}{\Omega_0}\{\gamma(\omega-|\mathbf{k}|v)\Omega_0\xi+\gamma|\mathbf{k}|v[1-\cos(\alpha-\theta)]$$
$$-\gamma|\mathbf{k}|v\ \sin\alpha\ \sin\theta[\sin\phi-\sin(\phi-\Omega_0\xi)]\}. \tag{11}$$

On setting the derivative of the phase (11) with respect to ϕ equal to zero, the condition for the stationary phase with respect to ϕ reduces to

$$\cos\phi=\cos(\phi-\Omega_0\xi). \tag{12}$$

One solution of Eq. (12) is $\phi=\phi_0+\Omega_0\tau=1/2\Omega_0\xi$. There is a second solution, which contributes equally in the following; this is taken into account simply by retaining only the one solution and multiplying the result by 2. On expanding about this point of stationary phase, Eq. (11) gives

$$k[X(\tau)-X(\tau-\xi)]$$
$$=\frac{1}{\Omega_0}\Bigg\{\gamma(\omega-|\mathbf{k}|v)\Omega_0\xi+\gamma|\mathbf{k}|v[1-\cos(\alpha-\theta)]$$
$$+\gamma|\mathbf{k}|v\ \sin\ \alpha\ \sin\ \theta\left[\frac{1}{2}\left(\phi-\frac{1}{2}\Omega_0\xi\right)^2+\Omega_0^2\xi^3/24\right]\Bigg\}. \tag{13}$$

The condition for stationary phase with respect to α is

$$\sin(\alpha-\theta)=\alpha_1,\quad \alpha_1=\frac{(\Omega_0\xi)^3}{24}\sin\theta\ \cos\theta, \tag{14}$$

where only the correction to first order in small quantities is retained. As with the gyrophase angle, there are two solutions that contribute equally, and one may retain only the solution $\alpha=\theta+\alpha_1$ and include another overall factor of 2 to take account of the other solution.

It is convenient to write

$$y=\Omega_0\xi,\quad a=\frac{\gamma(\omega-|\mathbf{k}|v)}{\Omega_0},\quad b=\frac{\gamma|\mathbf{k}|v\ \sin^2\theta}{8\Omega_0}, \tag{15}$$

and to change variables to $\delta\phi=\phi-\xi/2$, $\delta\alpha=\alpha-\theta-\alpha_1$. Then Eq. (13) reduces to

$$k[X(\tau)-X(\tau-\xi)]=ay+\frac{by^3}{3}+4by\,\frac{(\delta\alpha)^2}{\sin^2\theta}+4by\,\frac{\sin\alpha}{\sin\theta}\,(\delta\phi)^2. \tag{16}$$

In the integrand of Eq. (10) one makes the replacement $\Omega_0\tau=\delta\phi+\frac{1}{2}y,\ \alpha=\delta\alpha+\theta+\alpha_1$.

The specific integrals of the form (10) that appear in evaluating Eqs. (3) and (6) lead to the following replacements inside the integrands:

$$\langle(\delta\alpha)^2\rangle\to i\,\sin^2\theta\Delta,\quad \frac{\sin\alpha}{\sin\theta}\langle(\delta\phi)^2\rangle\to i\Delta,$$

$$\langle(\delta\alpha)^4\rangle\to-3\sin^4\theta\Delta^2,\quad \left(\frac{\sin\alpha}{\sin\theta}\right)^2\langle(\delta\phi)^4\rangle\to-3\Delta^2,$$

$$\Delta=\frac{1}{8by}. \tag{17}$$

The additional powers of y that appear in the denominator in view of $\Delta\propto y^{-1}$ lead to some y integrals that are dominated by the contribution from lower limit, $y=0$. Such terms are associated with the nonmagnetic part of the response, as may be seen by noting that at sufficiently high frequencies, the ratio Ω_0/ω becomes negligible, and the response must reduce to that of an unmagnetized plasma.

The method developed here should not be expected to give the unmagnetized part of the response accurately. (The model in which a fixed observer see pulses of radiation from a spiraling charge is not applicable to an unmagnetized system.) It follows that the magnetized part of the response, which is of interest here, may be isolated by subtracting the unmagnetized part, $\alpha_0^{\mu\nu}(k)$,

$$\alpha_{\rm mag}^{\mu\nu}(k)=\alpha^{\mu\nu}(k)-\alpha_0^{\mu\nu}(k). \tag{18}$$

The transverse part of the high frequency $(k^2\approx 0)$ response of any unmagnetized plasma reduces to [7]

$$\alpha_0^{\mu\nu}\approx-\frac{q^2}{m}\int d^4pF(p)g^{\mu\nu}=-\omega_{p0}^2\,g^{\mu\nu}, \tag{19}$$

with $\mu,\nu=1{,}2$, and where ω_{p0} is the proper plasma frequency.

The components of the response tensor may be expressed in terms of integrals of the form

$$I^{(n)}(a,b)=\int_0^\infty dy\;y^n\exp\left[iay+i\frac{1}{3}by^3\right]. \tag{20}$$

Some relevant properties of these functions are discussed in the next section.

For the forward-scattering form (3) the result is

$$\alpha^{\mu\nu}(k)=-\frac{q^2\phi(\theta)\Omega_0}{m\omega}\int d\gamma\,\frac{N(\gamma)}{\gamma^2v}\,J^{\mu\nu}(a,b), \tag{21}$$

$$J^{11}(a,b)=\tfrac{4}{3}\,bI^{(1)}(a,b)-a,$$

$$J^{22}(a,b)=4bI^{(1)}(a,b)+\tfrac{8}{3}\,ia^2I^{(0)}(a,b)+\tfrac{5}{3}\,a,$$

$$J^{12}(a,b)=-\tfrac{1}{2}\,\eta\,\cos\theta\{[\tfrac{16}{9}\,ia^2I^{(1)}(a,b)-\tfrac{20}{9}\,aI^{(0)}(a,b)-2iI^{(-1)}(a,b)+2i]+g(\theta)[-\tfrac{4}{3}\,aI^{(0)}(a,b)-2iI^{(-1)}(a,b)+i\,\tfrac{1}{3}]\}, \tag{22}$$

with $J^{21}(a,b)=-J^{12}(a,b)$ and with $g(\theta)=\tan\theta\,\phi'(\theta)/\phi(\theta)$. For the Vlasov form (6) the result is

$$\alpha^{\mu\nu}(k)=\alpha_{(\gamma)}^{\mu\nu}(k)+\alpha_{(\alpha)}^{\mu\nu}(k), \tag{23}$$

$$\alpha_{(\gamma)}^{\mu\nu}(k)=-\frac{q^2\phi(\theta)\Omega_0}{m\omega}\int d\gamma\;\gamma v\,\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2v}\right]h^{\mu\nu},$$

$$\alpha_{(\alpha)}^{12}(k)=\frac{q^2\phi(\theta)\Omega_0}{m\omega}\int d\gamma\,\frac{N(\gamma)}{\gamma}\,(\tfrac{1}{2}\,\eta\,\cos\theta)[-g(\theta)iI^{(-1)}(a,b)],$$

$$\begin{pmatrix}h^{11}\\ h^{22}\\ h^{12}\end{pmatrix}=\begin{pmatrix}-aI^{(-1)}(a,b)-bI^{(1)}(a,b)\\ -aI^{(-1)}(a,b)-3bI^{(1)}(a,b)\\ \tfrac{1}{2}\,\eta\,\cos\theta[(2+g(\theta))iI^{(-1)}(a,b)+\tfrac{4}{3}\,aI^{(0)}(a,b)-i\,\tfrac{4}{3}]\end{pmatrix}, \tag{24}$$

with $h^{21}=-h^{12}$. Partially integrating the form (23) with respect to γ (assuming $a\propto 1/\gamma$ and $b\propto\gamma$) reproduces Eq. (21) with Eq. (22). However, some care needs to be taken with the constant terms in $J^{11}(a,b)$ and $J^{22}(a,b)$; these terms are associated with the limit $y=0$, and hence with the unmagnetized part of the response. The unmagnetized part is not given correctly by either Eq. (21) with Eq. (22) or by Eq. (23) with Eq. (24). (They are incorrect by factors of 2 and -2, respectively, in the asymptotic limit considered below.) This inconsistency may be attributed to an inadequate treatment of the singular terms in the limit $y\rightarrow 0$. This inconsistency may be avoided by subtracting the unmagnetized part of the response using Eq. (18), and it is then straightforward to treat the unmagnetized part of the response using the theory for an unmagnetized plasma.

IV. PROPERTIES OF THE RESPONSE FUNCTIONS

The response tensor in the forms (21) and (23) involve three plasma dispersion functions, $I^{(n)}(a,b)$ with $n=-1$, 0, 1. These functions are defined by Eq. (20) for $n\geqslant 0$. For negative n, the definition (20) may be modified so that the functions are nonsingular by integrating the relation $dI^{(n)}(a,b)/da=iI^{(n+1)}(a,b)$ between 0 and a to generate the function for $n=-1$ from that for $n=0$, and so on.

These functions satisfy a recursion relation

$$bI^{(n+3)}(a,b)=-aI^{(n+1)}(a,b)+i(n+1)I^{(n)}(a,b) \quad \text{for } n\neq -1,$$

$$bI^{(2)}(a,b)=-aI^{(0)}(a,b)+i, \tag{25}$$

which follows by partially integrating in Eq. (20). The recursion relation (25) is used to reexpress all the $I^{(n)}(a,b)$ in terms of $I^{(-1)}(a,b)$, $I^{(0)}(a,b)$, and $I^{(1)}(a,b)$ in Eqs. (22) and (24).

For $n\leqslant 0$, $I^{(n)}(a,b)$ as defined by Eq. (20) would be singular, due to the divergence at $y=0$. The divergent terms are associated with the unmagnetized part of the response. These terms are subtracted using Eq. (18). It is then convenient to redefine $I^{(n)}(a,b)$ for $n<0$ so that all the functions are nonsingular. This may be achieved by integrating the relation $dI^{(n)}(a,b)/da=iI^{(n+1)}(a,b)$, between 0 and a. This gives

$$I^{(-1)}(a,b)=\int_0^\infty \frac{dy}{y}\,\{\exp[iay+i\,\tfrac{1}{3}\,by^3]-1\} \tag{26}$$

for $n=-1$, and a further such integration may be used to define $I^{(n)}(a,b)$ for $n=-2$.

The functions $I^{(n)}(a,b)$ are causal functions. This follows from the fact that the parameter a is proportional to the frequency ω, and y is a linear function of (proper) time, so that Eq. (20) defines $I^{(n)}(a,b)$ as the temporal Fourier transform of a function that vanishes for negative times. Hence $I^{(n)}(a,b)$ must satisfy the Kramers-Kronig relations

$$\begin{pmatrix} \mathrm{Im}I^{(n)}(a,b) \\ \mathrm{Re}I^{(n)}(a,b) \end{pmatrix}=\frac{1}{\pi}\mathcal{P}\int \frac{da'}{a'-a}\begin{pmatrix} \mathrm{Re}I^{(n)}(a',b) \\ -\mathrm{Im}I^{(n)}(a',b) \end{pmatrix}, \tag{27}$$

where $\mathcal{P}$ denotes the Cauchy principal value.

The real and imaginary parts of the function $I^{(0)}(a,b)$ are related to the Airy functions [8]

$$\mathrm{Ai}(z)=\frac{1}{\pi}\int_0^\infty dt\,\cos(zt+\tfrac{1}{3}t^3),$$
$$\mathrm{Gi}(z)=\frac{1}{\pi}\int_0^\infty dt\,\sin(zt+\tfrac{1}{3}t^3), \tag{28}$$

respectively. One has

$$I^{(0)}(a,b)=\pi b^{-1/3}[\mathrm{Ai}(z)+i\mathrm{Gi}(z)],$$

$$I^{(1)}(a,b)=-i\pi b^{-1/3}[\mathrm{Ai}'(z)+i\mathrm{Gi}'(z)],$$

$$I^{(-1)}(a,b)=i\pi\int_0^z dz'[\mathrm{Ai}(z')+i\mathrm{Gi}(z')],\quad z=a/b^{1/3}. \tag{29}$$

The parts that involve the function $\mathrm{Ai}(z)$ contribute to the anti-Hermitian part of the response tensor, and the parts that involve the function $\mathrm{Gi}(z)$ contribute the Hermitian part of the response tensor.

The Airy function $\mathrm{Ai}(z)$ can be represented as a Bessel function of order 1/3, and in the case of relevance to synchrotron emission these are Macdonald functions. The relevant representations in terms of Bessel functions are

$$\mathrm{Re}I^{(0)}(a,b)=\frac{1}{\sqrt{3}}\left(\frac{a}{b}\right)^{1/2}K_{1/3}(R),$$

$$\mathrm{Im}I^{(1)}(a,b)=\frac{1}{\sqrt{3}}\,\frac{a}{b}\,K_{2/3}(R),$$

$$\mathrm{Im}I^{(-1)}(a,b)=-\frac{1}{\sqrt{3}}\int_R^\infty dt\,K_{1/3}(t), \tag{30}$$

with $R=2a^{3/2}/3b^{1/2}$.

V. SYNCHROTRON ABSORPTION

A check on the validity of this theory is that the anti-Hermitian part of $\alpha^{\mu\nu}(k)$ reproduces the known formulas for synchrotron absorption. The anti-Hermitian part of Eq. (23) is inserted in the absorption coefficient $\gamma^{\mu\nu}(k)=i(\mu_0/\omega\;d|\mathbf{k}|/d\omega)\alpha^{A\mu\nu}(k)$. Using Eq. (30) and the recursion relations for the Macdonald functions to write

$$-\int_R^\infty dt\ K_{1/3}(t)+K_{2/3}(R)=\int_R^\infty dt\ K_{5/3}(t)-K_{2/3}(R), \tag{31}$$

one finds

$$\begin{bmatrix}\gamma^{11}(k)\\ \gamma^{22}(k)\end{bmatrix}=-\frac{\sqrt{3}\mu_0 q^2\Omega_0\phi(\theta)\sin\theta}{8m\omega^2}\int_1^\infty d\gamma\ \gamma\zeta v^{1/2}\ \frac{df(\gamma m v)}{d\gamma}\left[R\int_R^\infty dt\ K_{5/3}(t)\mp K_{2/3}(R)\right], \tag{32}$$

$$\gamma^{12}(k)=-\frac{4\pi\sqrt{3}\mu_0 q^2\Omega_0\phi(\theta)\sin\theta}{8m\omega^2}\int_1^\infty d\gamma\ \gamma\zeta v^{1/2}\left(\frac{2i\eta\cos\theta}{3\zeta v^{1/2}\sin\theta}\right)\left\{\left[[2+g(\theta)]\int_R^\infty dt\ K_{1/3}(t)+2K_{1/3}(R)\right]\frac{df(\gamma m v)}{d\gamma}\right.$$
$$\left.-g(\theta)\int_R^\infty dt\ K_{1/3}(t)\ \frac{f(\gamma m v)}{\gamma}\right\}, \tag{33}$$

where the approximation $|\mathbf{k}|=\omega$ is assumed except in $\omega-|\mathbf{k}|v=\omega/2\zeta^2$. The result (32) for $v\rightarrow 1$ reproduces a well-known expression for the synchrotron absorption coefficient for the linearly polarized components [9] derived from the emission coefficient by appealing to detailed balance; Eq. (33) reproduces a known form for the small circularly polarized component of synchrotron absorption [10], which was also derived appealing to detailed balance.

VI. THE HERMITIAN PART OF THE RESPONSE TENSOR

Approximations available for Gi(z) are relevant to the evaluation of the Hermitian part. Some known results are summarized in Appendix B. The case of most interest is high frequencies, corresponding to $a\gg 1$. In this case the asymptotic expansions of the plasma dispersion functions give

$$I^{(0)}(a,b)=\frac{i}{a}\left(1+\frac{2b}{a^3}\right),\quad I^{(1)}(a,b)=-\frac{1}{a^2},$$
$$I^{(-1)}(a,b)=-\ln a, \tag{34}$$

with $a=\omega/2\Omega_0\gamma$. Omitting the terms associated with the unmagnetized part of the response, the leading contributions to the magnetized part of the response in Eq. (22) are

$$J^{11}(a,b)=-\tfrac{2}{3}\ \frac{\Omega_0}{\omega}\ \gamma^3\sin^2\theta,\quad J^{22}(a,b)=7J^{11}(a,b),$$
$$J^{12}(a,b)=\tfrac{1}{2}\ i\eta\ \cos\theta[1+g(\theta)]\ln\left(\frac{\omega}{2\Omega_0\gamma}\right), \tag{35}$$

where the logarithmic term is assumed to dominate in $J^{12}(a,b)$. The ratio 1 to 7 for the diagonal components of the magnetized part of the response is evidently a characteristic feature of the high-frequency response of a synchrotron-emitting gas.

Besides the requirement that the formulas for synchrotron absorption be reproduced, a further check on the validity of the results (21) and (23) is to use it to evaluate the response tensor for a relativistic thermal distribution. The result may then be compared with the appropriate limit of the exact expression for a thermal distribution of arbitrary temperature derived by Trubnikov [11]. The highly relativistic limit of Trubnikov's tensor is derived elsewhere [12], where it is shown that the two results agree, with the exception that the argument of the logarithm in the off-diagonal term is determined only to within a factor of order unity in either derivation.

The case of most interest in astrophysical plasmas is a power-law distribution,

$$N(\gamma)=\begin{cases}N_0\gamma^{-\beta} & \text{for}\quad \gamma_1<\gamma<\gamma_2\\ 0 & \text{otherwise,}\end{cases}$$

$$N_0=\begin{cases}n(\beta-1)/(\gamma_1^{1-\beta}-\gamma_2^{1-\beta})^{-1} & \text{for } \beta\neq 1,\\ n\ \ln(\gamma_2/\gamma_1) & \text{for } \beta=1,\end{cases} \tag{36}$$

where n is the number density of the relativistic particles. Inserting Eq. (36) into Eq. (21) with Eq. (35) gives, for $\beta>2$ and $\gamma_2\gg\gamma_1$,

$$\begin{pmatrix}\alpha^{11}(k)\\ \alpha^{22}(k)\end{pmatrix}=\frac{\omega_{p0}^2}{\mu_0}+\frac{2q^2n\phi(\theta)}{3m}\ \frac{\Omega_0^2}{\omega^2}\ \sin^2\theta\ \frac{\beta-1}{\beta-2}\ \gamma_1\begin{pmatrix}1\\ 7\end{pmatrix},$$

$$\alpha^{12}(k)=-\frac{i}{2}\ \eta\ \cos\theta\ [1+g(\theta)]\ \frac{q^2n\phi(\theta)}{m}\ \frac{\Omega_0}{\omega}\ \frac{\beta-1}{\beta-2}\ \frac{1}{\gamma_1}\ln\left(\frac{\omega}{\Omega_0\gamma_1\sin\theta}\right), \tag{37}$$

where the unmagnetized contribution (19) is now included. For $\beta<2$, the γ integral for the 11 and 22 components is dominated by the largest γ for which the approximations are valid. Assuming that the approximation (34) applies for $a>1$ and that the functions are negligible for $a<1$, the γ integral is cut off at $\gamma=(\omega/\Omega_0\sin\theta)^{1/2}$. Hence for $\beta<2$ and $\gamma_1<(\omega/\Omega_0\sin\theta)^{1/2}<\gamma_2$, Eq. (37) is replaced by

$$\alpha^{11}(k)=\frac{\omega_{p0}^2}{\mu_0}+\frac{2q^2n\phi(\theta)}{3m}\frac{\Omega_0^2}{\omega^2}\sin^2\theta\,\frac{\beta-1}{2-\beta}\times(\gamma_1^{1-\beta}-\gamma_2^{1-\beta})^{-1}\left(\frac{\omega}{\Omega_0\sin\theta}\right)^{(2-\beta)/2}, \quad (38)$$

with $\alpha^{22}(k)=7\alpha^{11}(k)$; the $\alpha^{12}(k)$ is unchanged. At higher frequencies, $(\omega/\Omega_0\sin\theta)^{1/2}>\gamma_2$, the frequency dependence of the diagonal terms reverts to that in Eq. (37).

The general form of the results (37) and (38) is consistent with those found by Sazonov [2], but the numerical coefficient of the diagonal terms are different.

The properties of the two natural wave modes (labeled $\pm$) of a plasma with response tensor of the form found here, cf. Eq. (37) or (38), include the dispersion relation

$$k^2=k_\pm^2=\frac{\mu_0}{2}\{\alpha^{11}+\alpha^{22}\pm[(\alpha^{11}-\alpha^{22})^2+4\alpha^{12}\alpha^{21}]^{1/2}\}, \quad (39)$$

where arguments (k) are omitted, and the polarization vectors are

$$e_\pm^\mu=\frac{T_\pm e^1+ie^2}{(1+T_\pm^2)^{1/2}},$$

$$T_\pm=\frac{\alpha^{11}-\alpha^{22}\mp[(\alpha^{11}-\alpha^{22})^2+4\alpha^{12}\alpha^{21}]^{1/2}}{2i\alpha^{12}}. \quad (40)$$

One has $|\alpha^{12}|\ll|\alpha^{11}-\alpha^{22}|$, except for very small $\sin\theta$, so that the natural modes are nearly linearly polarized. The effect of an admixture of cold plasma on the wave modes is discussed elsewhere [12].

VII. CONCLUSIONS

The main result of this paper is that the method of stationary phase may be used to treat the response of a highly relativistic electron gas. It is shown that the anti-Hermitian part of the resulting response tensor reproduces the known formulas for synchrotron absorption exactly. Moreover, unlike an earlier treatment [2], the present method preserves the causal properties of the relevant plasma dispersion functions, so that this check confirms the validity of the Hermitian part (to within terms whose Hilbert transform is zero). Another check on the validity of the results follows by evaluating the response tensor for a relativistic thermal distribution and comparing the result with the appropriate limit of Trubnikov's response tensor [12].

A technical difficulty arises in the theory presented here due to certain terms being singular. Mathematically, these appear through the integral $I^{(-1)}(a,b)=\int_0^\infty dy\; y^{-1}\exp[i(ay+1/3by^3)]$ and, physically, they are associated with the unmagnetized part of the response. This technical difficulty is avoided here by (a) subtracting the unmagnetized part, cf. Eq. (18), so that the remaining magnetized part has no singularities, (b) using unmagnetized theory to calculate the unmagnetized part of the response, and (c) redefining $I^{(-1)}(a,b)$, cf. Eq. (20), to remove the divergent term. Consistent results are then obtained for the diagonal terms. The off-diagonal term contains a logarithm associated with $I^{(-1)}(a,b)$, and this term is determined only to within a factor of order unity in the argument of the logarithm.

The high-frequency response is evaluated for the case of a power-law particle distribution, which is the case of most astrophysical interest. The result is similar in form to that obtained in [2], but differs from it by numerical factors. These numerical differences may be attributed to an incorrect relation between the Hermitian and anti-Hermitian parts in [2]. The natural wave modes of such a plasma are nearly linearly polarized, as found in [2] and contrary to a claim made in [3].

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APPENDIX A

The specific quantities that appear in the integrands in Eqs. (3) and (6), after expanding in powers of quantities of order $1/\gamma$, include

$$T^{\mu'\nu'}(\xi)=\frac{1}{\Omega_0}\begin{pmatrix}-y & -\frac{1}{2}\eta y^2\\ \frac{1}{2}\eta y^2 & -y\end{pmatrix},$$

$$k_\alpha k_\beta T^{\alpha\beta}(\xi)=\frac{|\mathbf{k}|^2\sin^2\theta}{\Omega_0}\frac{y^3}{6},$$

$$k_\beta T^{\mu'\beta}(\xi)=\frac{|\mathbf{k}|\sin\theta}{\Omega_0}(-\cos\theta\tfrac{1}{6}y^3,-\tfrac{1}{2}\eta y^2),$$

$$k_\alpha T^{\alpha\nu'}(\xi)=\frac{|\mathbf{k}|\sin\theta}{\Omega_0}(-\cos\theta\,\tfrac{1}{6}y^3,\tfrac{1}{2}\eta y^2),$$

$$u^{\mu'}(\tau)=\gamma v[\delta\alpha+\alpha_1-\tfrac{1}{2}\sin\theta(\delta\phi+\tfrac{1}{2}\xi)^2,$$

$$-\eta\sin\theta(\delta\phi+\tfrac{1}{2}\xi)],$$

$$ku(\tau)=\Omega_0\left\{a+4b\left[\frac{(\delta\alpha+\alpha_1)^2}{\sin^2\theta}+\frac{\sin\alpha}{\sin\theta}(y+\delta\phi)^2\right]\right\},$$

$$ka(\tau)=\gamma|\mathbf{k}|v\,\delta\alpha,\phi(\alpha)=\phi(\theta)\left[1+(\alpha-\theta)g(\theta)\frac{\cos\theta}{\sin\theta}\right],$$

$$\sin\alpha=\sin\theta\left[1+(\alpha-\theta)\frac{\cos\theta}{\sin\theta}\right]. \quad (A1)$$

Also, the term $ku(\tau)\widetilde{u}^\nu$ in Eq. (9) does not contribute to the transverse component; after expanding in $1/\gamma$, the first term in $ku(\tau)a^{\nu}(\tau)-ka(\tau)u^{\nu}(\tau)$ is negligible; and to lowest order $ka(\tau)$ does not depend on τ.

APPENDIX B: APPROXIMATIONS TO HERMITIAN PART

The approximations available for $\mathrm{Gi}(z)$ are for large and small z. The leading terms in the asymptotic expansion for $z \gg 1$ are [13]

$$\mathrm{Gi}(z) \sim \frac{1}{\pi}\left(\frac{1}{z}+\frac{2}{z^4}+\cdots\right), \quad \mathrm{Gi}'(z) \sim \frac{1}{\pi}\left(-\frac{1}{z^2}+\cdots\right), \tag{B1}$$

$$\int^z dz' \,\mathrm{Gi}(z') \sim \frac{1}{\pi}\left(\ln z+\frac{2\gamma+\ln 3}{3}-\frac{2}{3z^3}+\cdots\right),$$

where $\gamma = 0.577,\ldots$ is Euler's constant. The expansion for $z \ll 1$ gives [11]

$$\mathrm{Gi}(z) = \frac{1}{\pi}\left[\frac{3^{1/3}}{2}\,\Gamma(4/3)+\frac{3^{2/3}}{4}\,\Gamma(5/3)\;z-\frac{z^2}{2}+\cdots\right], \tag{B2}$$

with [8] $\mathrm{Gi}(0)=0.205$, $\mathrm{Gi}'(0)=0.149$.

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